W48. Let ABC a triangle such that

$$s^2 = 2R^2 + 8Rr + 3r^2$$

Then $\frac{R}{r} = 2$ or $\frac{R}{r} \geq \sqrt{2} + 1$.

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First note that $s^2 = 2R^2 + 8Rr + 3r^2 \iff a^2 + b^2 + c^2 = 4(R + r)^2$.

Indeed, since* $ab + bc + ca = s^2 + 4Rr + r^2$ and $s^2 = 2R^2 + 8Rr + 3r^2$ then

$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) =$

$4s^2 - 2(s^2 + 4Rr + r^2) = 2(s^2 - 4Rr - r^2) = 2(2R^2 + 8Rr + 3r^2 - 4Rr - r^2) = 4(R + r)^2$.

If $\triangle ABC$ isn't acute triangle then** $a^2 + b^2 + c^2 \leq 8R^2$ and, therefore,

$8R^2 \geq 4(R + r)^2 \iff 8R^2 - 4(R + r)^2 \geq 0 \iff R^2 - 2Rr - r^2 \geq 0 \iff \frac{R}{r} \geq 1 + \sqrt{2}$.

Let $\triangle ABC$ is acute. Then $a^2 + b^2 + c^2 \geq 4(R + r)^2$ and equality holds iff

$a = b = c$.

Indeed, we have $a^2 + b^2 + c^2 \geq 4(R + r)^2 \iff 4R^2(\sin^2 A + \sin^2 B + \sin^2 C) \geq$

$4R^2\left(1 + \frac{r}{R}\right)^2 \iff \sin^2 A + \sin^2 B + \sin^2 C \geq (\cos A + \cos B + \cos C)^2 \iff$

$\sum_{cyc}(1 - \cos^2 A) \geq \sum_{cyc}\cos^2 A + 2\sum_{cyc}\cos B\cos C \iff \sum_{cyc}(\cos A + \cos B)^2 \leq 3$.

Since by Cauchy Inequality $(\cos A + \cos B)^2 \leq (a\cos B + b\cos A)\left(\frac{\cos B}{a} + \frac{\cos A}{b}\right) =$

$c\left(\frac{\cos B}{a} + \frac{\cos A}{b}\right)$ then $\sum_{cyc}(\cos A + \cos B)^2 \leq \sum_{cyc}\left(\frac{c\cos B}{a} + \frac{c\cos A}{b}\right) =$

$\frac{c\cos B}{a} + \frac{c\cos A}{b} + \frac{a\cos C}{b} + \frac{a\cos B}{c} + \frac{b\cos A}{c} + \frac{b\cos C}{a} =$

$\sum_{cyc}\left(\frac{c\cos B + b\cos C}{a}\right) = \sum_{cyc}\frac{a}{a} = 3,$ where equality holds iff $\frac{\sqrt{a\cos B}}{\sqrt{\frac{\cos B}{a}}} = \frac{\sqrt{b\cos A}}{\sqrt{\frac{\cos A}{b}}} \iff$

$a = b$ and cyclic $b = c, c = a,$ that is iff $a = b = c \iff R = 2r \iff \frac{R}{r} = 2$.

Thus, for acute triangle $s^2 = 2R^2 + 8Rr + 3r^2 \iff a^2 + b^2 + c^2 = 4(R + r)^2 \iff \frac{R}{r} = 2$.

* $sr^2 = (s - a)(s - b)(s - c) \iff sr^2 = s^3 - 2s^3 + s(ab + bc + ca) - abc \iff$

$sr^2 = -s^3 + s(ab + bc + ca) - 4Rrs \iff r^2 = -s^2 + ab + bc + ca - 4Rr \iff$

$ab + bc + ca = s^2 + 4Rr + r^2$.

** Let $C \geq \pi/2$. Since $a = 2R\sin A, b = 2R\sin B, c = 2R\sin C$ and $A, B < \pi/2$

then $8R^2 - (a^2 + b^2 + c^2) = 2R^2(4 - (2\sin^2 A + 2\sin^2 B + 2\sin^2 C)) =$

$2R^2(1 + \cos 2A + \cos 2B + \cos 2C) = 8R^2\cos(A + B)\cos(B + C)\cos(C + A) =$

$8R^2\cos A\cos B(-\cos C) \geq 0$.